# Linear differential equations

$$\frac{dy}{dx} + Py = Q$$

P & Q either Constant or f' of linear diff. Eq. order 1 is y

**Step-1** I.F → Integrating factor

$$I.F = e^{\int P dx}$$

**Step-2** $y \times (I.F) = \int Q(I.F)dx + C$

$$\frac{dx}{dy} + Px = Q$$

P & Q either Constant or f' of linear diff. Equ. order 1 is x

**Step-1** I.F = Integrating factor

$$I.F = e^{\int P dy}$$

**Step-2** $x \times I.F = \int Q(I.F)\,dy + C$

# Exercise 9.7

1. Which of the following equations are linear differential equations of first order in y?

i) $\frac{dy}{dx} + 3y = \sin x$

It is L·D·E in order = 1

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dx}{dy} + Px = Q$$

ii) $x^2 \frac{dy}{dx} - 2xy = x^3 + 2x^2 - 5$

$$\frac{dy}{dx} - \frac{2xy}{x^2} = \frac{x^3 + 2x^2 - 5}{x^2}$$

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dy}{dx} - \frac{2y}{x} = \frac{x^3 + 2x^2 - 5}{x^2}$$

$$P = -\frac{2}{x} \qquad Q = \frac{x^3 + 2x^2 - 5}{x^2}$$

It is L·D·E in order = 1

iii) $x \log x dy + (3y - 5 \log x) dx = 0$

$$x \log x \, dy = (5 \log x - 3y) \, dx$$

$$x \log x \frac{dy}{dx} = 5 \log x - 3y$$

$$\frac{dy}{dx} = \frac{5}{x} - \frac{3y}{x \log x}$$

$$\frac{dy}{dx} + \frac{3}{x \log x} y = \frac{5}{x}$$

$$P = \frac{3}{x \log x}$$

$$Q = \frac{5}{x}$$

$$\frac{dy}{dx} + Py = Q$$

It is L·D·E in order = 1

iv) $y\frac{dy}{dx} + 5x = y\cos x$

$$\frac{dy}{dx} + \frac{5}{y}x = \cos x$$

It is not L·D·E in order = 1

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dx}{dy} + Px = Q$$

2.Which of the following equations are linear differential equations of first order in x?

i) $\frac{dx}{dy} - \frac{2x}{y} = 3y^3 - 5y + 1$

$$\frac{dx}{dy} + Px = Q$$

$$\frac{dx}{dy} - \frac{2}{y}x = 3y^3 - 5y + 1$$

$$P = -\frac{2}{y} \qquad Q = 3y^3 - 5y + 1$$

It is L·D·E in order = 1

ii) $(\cos y)\frac{dx}{dy} - 2x\sin y = \cos^3 y$

$$\frac{dx}{dy} - 2x\frac{\sin y}{\cos y} = \frac{\cos^3 y}{\cos y}$$

$$\frac{dx}{dy} - 2x\tan y = \cos^2 y$$

$$\frac{dx}{dy} + Px = Q$$

$P = -2\tan y \quad Q = \cos^2 y$

It is L.D.E of order = 1

iii) $(1+y^2)dx = (\tan^{-1}y - 2x)dy$

$$1+y^2\,dx = (\tan^{-1}y - 2x)\,dy$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{2x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{2x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\frac{dx}{dy} + Px = Q$$

l·D·E in order = 1

$$P = \frac{2}{1+y^2} \qquad Q = \frac{\tan^{-1}y}{1+y^2}$$

iv) $x^2 \frac{dx}{dy} + 2x^3 y = 3y^2 - 2$

$$\frac{dx}{dy} + Px = Q$$

$$x^2 \frac{dx}{dy} + 2x^3 y = 3y^2 - 2$$

$$\frac{dx}{dy} + 2xy = \frac{3y^2}{x^2} - \frac{2}{x^2}$$

$$\frac{dx}{dy} + 2xy =$$

It is not L.D.E in order = 1

3.Write an integrating factors of each of the following linear differential equation

i) $x\frac{dy}{dx} = 2x^2 + y\,, x > 0$

$$\frac{dy}{dx} = \frac{2x^2}{x} + \frac{y}{x}$$

$$\frac{dy}{dx} - \frac{y}{x} = 2x$$

$P = -\frac{1}{x}$ $\quad Q = 2x$

$I.F = e^{\int P\,dx}$

$= e^{\int -\frac{1}{x}\,dx}$

$= e^{-\log x}$

$= e^{\log x^{-1}}$

$I.F = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}$

$\frac{dy}{dx} + Py = Q$

$I.F = e^{\int P\,dx}$

$\frac{dx}{dy} + Px = Q$

$I.F = e^{\int P\,dy}$.

ii) $2x\frac{dy}{dx} + y = 6x^3 + 5\,, x > 0$

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{6x^3}{2x} + \frac{5}{2x}$$

$\log m^n = n \log m$

$$P = \frac{1}{2x}$$

$$I.F = e^{\int P\,dx}$$

$$= e^{\int \frac{1}{2x}\,dx}$$

$$I.F = e^{\frac{1}{2}\log x} = e^{\log x^{1/2}} = e^{\log \sqrt{x}} = \sqrt{x}$$

iii) $\frac{dy}{dx} + 2y\tan x = \sin x$

$\frac{dy}{dx} + Py = Q$

$P = 2\tan x \qquad Q = \sin x$

$I.F = e^{\int P\,dx} = e^{2\int \tan x\,dx}$

$= e^{-2\log \cos x}$

$= e^{\log \cos^{-2} x}$

$= e^{\log \frac{1}{\cos^2 x}}$

$I.F = e^{\log \sec^2 x} = \sec^2 x$

iv) $x\frac{dy}{dx} + y - xy\cot x = 0$

$$\frac{dy}{dx} + \frac{y}{x} - y\cot x = 0$$

$$\frac{dy}{dx} + \left(\frac{1}{x} - \cot x\right)y = 0$$

$$P = \frac{1}{x} - \cot x \qquad Q = 0$$

$$I.F = e^{\int P\,dx}$$

$$= e^{\int \left(\frac{1}{x} - \cot x\right)dx}$$

$$= e^{\log x - \log \sin x}$$

$$= e^{\log \frac{x}{\sin x}} = \frac{x}{\sin x}$$

4.) Write an integrating factors of each of the following linear differential equation:

i) $ydx - (x + 2y^2)dy = 0\,, y > 0$

$$y\,dx = (x + 2y^2)dy$$

$$y\frac{dx}{dy} = (x + 2y^2)$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

$$\boxed{\frac{dx}{dy} + Px = Q}$$

$$P = -\frac{1}{y} \qquad Q = 2y$$

$$I.F = e^{\int P\,dy}$$

$$= e^{-\int \frac{1}{y}\,dy} = e^{-\log y} = e^{\log y^{-1}}$$

$$= e^{\log\left(\frac{1}{y}\right)}$$

$$\boxed{I.F = \frac{1}{y}}$$

ii) $ye^y dx = (y^3 + 2xe^y)dy$ , $y \neq 0$

$$\frac{dx}{dy} = \frac{y^3}{ye^y} + \frac{2xe^y}{ye^y}$$

$$\frac{dx}{dy} = \frac{y^2}{e^y} + \frac{2x}{y}$$

$$\boxed{\frac{dx}{dy} - \frac{2x}{y} = \frac{y^2}{e^y}}$$

$$I.F = e^{\int P\,dy}$$

$$= e^{-2\int \frac{1}{y}\,dy} = e^{-2\log y}$$

$$= e^{\log y^{-2}} = e^{\log\left(\frac{1}{y^2}\right)}$$

$$\boxed{I.F = \frac{1}{y^2}}$$

iii) $dx + xdy = (e^{-y}\log y)dy$ ,y>0

$dx = dy\,(e^{-y}\log y - x)$

$\frac{dx}{dy} = e^{-y}\log y - x$

$\frac{dx}{dy} + x = e^{-y}\log y$

$P = 1 \quad Q = e^{-y}\log y$

$I.F = e^{\int P\,dy}$

$= e^{\int 1\cdot dy} = e^{y}$ Ans

iv) $(1-y^2)\frac{dx}{dy} + yx = ay\ ,\ |y| < 1$

$$\frac{dx}{dy} + \frac{yx}{1-y^2} = \frac{ay}{1-y^2}$$

$$P = \frac{y}{1-y^2}$$

$$I.F = e^{\int P\,dy} = e^{\int \frac{y}{1-y^2}\,dy}$$

$1-y^2 = t$

$-2y\,dy = dt$

$y\,dy = -\frac{1}{2}dt$

$$= e^{-\frac{1}{2}\int \frac{1}{t}\,dt} = e^{-\frac{1}{2}\log t} = e^{-\frac{1}{2}\log(1-y^2)} = e^{\log(1-y^2)^{-1/2}}$$

$$I.F = e^{\log \frac{1}{\sqrt{1-y^2}}} = \frac{1}{\sqrt{1-y^2}}$$

Solve the following (5 to 19) differential equations:

5 i) $\frac{dy}{dx} + \frac{y}{x} = x^2$

Step-1 $P = \frac{1}{x}$ $\quad Q = x^2$

$$I.F = e^{\int P\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$$

$$\boxed{I.F = x}$$

Step-2

$$y \times I.F = \int I.F\,(Q)\,dx + c$$

$$xy = \int x \times x^2\,dx + c$$

$$xy = \int x^3\,dx + c$$

$$\boxed{xy = \frac{x^4}{4} + c}$$

$$y = \frac{x^3}{4} + \frac{c}{x}$$

ii) $x\frac{dy}{dx} + 2y = x^2$

$$\frac{dy}{dx} + \frac{2y}{x} = x$$

$$Q = x \quad P = \frac{2}{x}$$

$$I.F = e^{\int P\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\log x} = e^{\log x^2} = x^2$$

$$y \times I.F = \int I.F \times Q\,dx + C$$

$$yx^2 = \int x^2 \cdot x\,dx + C$$

$$y x^2 = \int x^3 \, dx + c$$

$$y x^2 = \frac{x^4}{4} + c$$

$$\boxed{y = \frac{x^2}{4} + \frac{c}{x^2}}$$

6.i) $\frac{dy}{dx} + 2y = 6e^x$

$P = 2 \qquad Q = 6e^x$

$I.F = e^{\int P\,dx} = e^{2\int 1\cdot dx} = e^{2x}$

$\boxed{I.F = e^{2x}}$

$y \times I.F = \int I.F \times Q\,dx + c$

$ye^{2x} = \int e^{2x} \cdot 6e^x\,dx + c$

$ye^{2x} = 6\int e^{3x}\,dx + c$

$ye^{2x} = \frac{6e^{3x}}{3} + c$

$\boxed{ye^{2x} = 2e^{3x} + c}$

$y = \frac{2e^{3x}}{e^{2x}} + \frac{c}{e^{2x}}$

$\boxed{y = 2e^x + ce^{-2x}}$

ii) $\frac{dy}{dx} + 3y = e^{-2x}$

$P = 3 \quad Q = e^{-2x}$

$I.F = e^{\int P\,dx} = e^{3\int 1\cdot dx} = e^{3x}$

$y \times I.F = \int I.F \times Q\,dx + c$

$y\,e^{3x} = \int e^{3x} \cdot e^{-2x}\,dx + c$

$ye^{3x} = \int e^{x}\,dx + c$

$ye^{3x} = e^{x} + c$

$y = \frac{e^{x}}{e^{3x}} + \frac{c}{e^{3x}}$

$$\boxed{y = e^{-2x} + ce^{-3x}}$$

7.i) $(1+x^2)\frac{dy}{dx} = 4x^2 - 2xy$

$$\frac{dy}{dx} = \frac{4x^2}{1+x^2} - \frac{2x}{1+x^2}y$$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

$$P = \frac{2x}{1+x^2} \qquad Q = \frac{4x^2}{1+x^2}$$

$$I.F = e^{\int P\,dx}$$

$$= e^{\int \frac{2x}{1+x^2}\,dx} = e^{\log(1+x^2)}$$

$$I.F = 1+x^2$$

$$y \times I.F = \int I.F \times Q\,dx + c$$

$$y \times (1+x^2) = \int (1+x^2) \times \frac{4x^2}{1+x^2}\,dx + c$$

$$y \times (1+x^2) = 4\int x^2\,dx + c$$

$$\boxed{y \times (1+x^2) = 4\frac{x^3}{3} + c}$$

ii) $\frac{dy}{dx} + \frac{y}{x} = e^{-x}$

$P = \frac{1}{x} \quad Q = e^{-x}$

$I.F = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$y \times I.F = \int I.F \times Q \, dx + c$

$xy = \int x \times e^{-x} dx + c$

$xy = -x e^{-x} + \int 1 \cdot e^{-x} dx + c$

$xy = -x e^{-x} - e^{-x} + c$

$$y = -e^{-x} - \frac{e^{-x}}{x} + \frac{c}{x}$$

8.i) $(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$

$$\frac{dy}{dx} + \frac{2xy}{x^2+1} = \frac{\sqrt{x^2+4}}{x^2+1}$$

$$P = \frac{2x}{x^2+1} \qquad Q = \frac{\sqrt{x^2+4}}{x^2+1}$$

$$I.F = e^{\int P\,dx} = e^{\int \frac{2x}{x^2+1}\,dx} = e^{\log(x^2+1)} = x^2+1$$

$$y \times I.F = \int I.F \times Q \, dx + C$$

$$y \times (x^2+1) = \int \cancel{x^2+1} \times \frac{\sqrt{x^2+4}}{\cancel{x^2+1}} \, dx + C$$

$$y(x^2+1) = \int \sqrt{x^2 + (2)^2} \, dx + C$$

$$y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2} \log \left| x + \sqrt{x^2+4} \right| + C$$

$$\boxed{y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + 2 \log \left| x + \sqrt{x^2+4} \right| + C}$$

ii) $x\frac{dy}{dx} + y = 3x\cos 2x\ , x > 0$

9.i) $x\frac{dy}{dx} - y = (x-1)e^x, x > 0$

Both Side dividing by $x$

$$\frac{dy}{dx} - \frac{y}{x} = \left(\frac{x-1}{x}\right)e^x$$

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dy}{dx} - \frac{y}{x} = \left(1 - \frac{1}{x}\right)e^x$$

$$P = -\frac{1}{x} \quad Q = \left(1 - \frac{1}{x}\right)e^x$$

$$I.F = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x}$$

$$y \times I.F = \int Q \cdot I.F \, dx + c$$

$$y \times \frac{1}{x} = \int \left(1 - \frac{1}{x}\right) e^x \times \frac{1}{x} \, dx + c$$

$$\frac{y}{x} = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) e^x \, dx + c$$

$$= \int \underset{\text{I}}{\frac{1}{x}} \underset{\text{II}}{e^x} \, dx - \int \frac{e^x}{x^2} \, dx + c$$

$$= \frac{1}{x}\int e^x dx - \int \left(\frac{d}{dx}\frac{1}{x}\int e^x dx\right) dx - \int \frac{e^x}{x^2} dx + c$$

$$= \frac{e^x}{x} + \int \frac{1}{x^2} e^x dx - \int \frac{e^x}{x^2} dx + c$$

$$\boxed{\frac{y}{x} = \frac{e^x}{x} + c}$$

$$\frac{y}{x} = \frac{e^x + cx}{x}$$

$$\boxed{y = e^x + cx}$$

ii) $\frac{dx}{dy} + x = \tan y + sec^2 y$

$\frac{dx}{dy} + Px = Q$

$P = 1 \qquad Q = \tan y + \sec^2 y$

$I.F = e^{\int P\,dy} = e^{\int 1\,dy} = e^{y}$

$x \times I.F = \int Q \cdot (I.F)\,dy + c$

$x\,e^{y} = \int (\tan y + \sec^2 y)\,e^{y}$

$= \int \tan y\, e^{y}\,dy + \int e^{y} \sec^2 y\,dy$

$= \tan y\, e^{y} - \int \left(\frac{d}{dy}\tan y \int e^{y} dy\right) dy + \int e^{y} \sec^2 y\,dy$

$= \tan y\, e^{y} - \int \sec^2 y\, e^{y}\,dy + \int e^{y} \sec^2 y\,dy$

$x\,e^{y} = \tan y\, e^{y} + c$

10.i) $x\frac{dy}{dx} + 2y = x\cos x$

$x\frac{dy}{dx} + 2y = x\cos x$

$\frac{dy}{dx} + 2\frac{y}{x} = \cos x$

$P = \frac{2}{x} \quad Q = \cos x$

$I.F = e^{\int P\,dx} = e^{2\int \frac{1}{x}dx} = e^{2\log x} = x^2$

$y \times I.F = \int Q \cdot (I.F)\,dx + c$

$yx^2 = \int x^2 \cos x\,dx + c$

$yx^2 = x^2 \sin x - \int \left(\frac{d}{dx}x^2 \int \cos x\,dx\right) dx$

$yx^2 = x^2 \sin x - \int 2x \sin x\,dx + c$

$yx^2 = x^2 \sin x - 2\left\{-x\cos x + \int \cos x\right\} + c$

$yx^2 = x^2 \sin x + 2x\cos x - 2\sin x + c$

ii) $(\sin x)\frac{dy}{dx} + (\cos x)y = \cos x \sin^2 x$

$$\frac{dy}{dx} + \left(\frac{\cos x}{\sin x}\right)y = \frac{\cos x \sin^2 x}{\sin x}$$

$$\frac{dy}{dx} + (\cot x)y = \sin x \cos x$$

$P = \cot x \quad Q = \sin x \cos x$

$$I.F = e^{\int P dx} = e^{\int \cot x \, dx} = e^{\log \sin x} = \sin x$$

$$y \times I.F = \int Q \times I.f \, dx + c$$

$$y \sin x = \int \sin x \cos x \cdot \sin x \, dx + c$$

$$y \sin x = \int \sin^2 x \cos x \, dx + c$$

$$y \sin x = \int t^2 \, dt + c \qquad \sin x = t$$

$$\cos x \, dx = dt$$

$$y \sin x = \frac{t^3}{3} + c$$

$$\boxed{y \sin x = \frac{\sin^3 x}{3} + c}$$

iii) $\sin x \frac{dy}{dx} - y = \sin x \tan \frac{x}{2}$

$\frac{dy}{dx} - \frac{y}{\sin x} = \tan \frac{x}{2}$

$P = -\frac{1}{\sin x} \qquad Q = \tan \frac{x}{2}$

$I.F = e^{\int P dx} = e^{-\int \frac{1}{\sin x} dx} = e^{-\int \operatorname{cosec} x \, dx} = e^{-\log \tan \frac{x}{2}}$

$I.f = e^{\log \cot \frac{x}{2}} = \cot \frac{x}{2}$

$y \cot \frac{x}{2} = \int \tan \frac{x}{2} \times \cot \frac{x}{2} dx + c$

$$y \cot\frac{x}{2} = \int 1 \cdot dx + c$$

$$\boxed{y \cot\frac{x}{2} = x + c}$$

11.i) $\frac{dy}{dx} = -\frac{x+y\cos x}{1+\sin x}$

$$\frac{dy}{dx} = -\frac{x}{1+\sin x} - y\frac{\cos x}{1+\sin x}$$

$$\frac{dy}{dx} + y\frac{\cos x}{1+\sin x} = \frac{-x}{1+\sin x}$$

$$P = \frac{\cos x}{1+\sin x} \qquad Q = \frac{-x}{1+\sin x}$$

$$I.F = e^{\int P\,dx}$$

$$= e^{\int \frac{\cos x}{1+\sin x}\,dx}$$

$$= e^{\log(1+\sin x)}$$

$$I.F = 1+\sin x$$

$$y(1+\sin x) = \int \frac{-x}{1+\sin x} \times (1+\sin x)$$

$$\boxed{y(1+\sin x) = -\left[\frac{x^2}{2}\right] + c}$$

ii) $\frac{dy}{dx} + y\cos x = e^{\sin x}\cos x$

$\frac{dy}{dx} + y\cos x = -\sin x$

$P = \cos x \quad Q = -\sin x$

$I.F = e^{\int P\,dx} = e^{\int \cos x\,dx} = e^{\sin x}$

$y \times e^{\sin x} = \int e^{\sin x} e^{\sin x}\cos x\,dx$

$= \int e^{2\sin x}\cos x\,dx$

$\sin x = t$

$y \times e^{\sin x} = \int e^{2t}\,dt$

$y \times e^{\sin x} = \frac{e^{2t}}{2} + c$

$$y \times e^{\sin x} = \frac{e^{2\sin x}}{2} + c$$

12. (i) $\frac{dy}{dx} - \tan x\, y = e^{\sin x}$

$\frac{dy}{dx} + Py = Q$

$P = -\tan x \qquad Q = e^{\sin x}$

$I.F = e^{\int P\,dx} = e^{-\int \tan x\,dx + \log \cos x} = e^{\log \cos x} = \cos x$

$I.F = \sec x$

$y \times \cos x = \int e^{\sin x} \cos x\, dx$

$\sin x = t$

$\cos x\, dx = dt$

$y \cos x = \int e^t dt$

$y \cos x = e^t + c$

$y \cos x = e^{\sin x} + c$

$$\frac{dy}{dx} + y = \cos x - \sin x$$

$P = 1 \qquad Q = \cos x - \sin x$

$$I.F = e^{\int P\,dx} = e^{\int 1\cdot dx} = e^{x}$$

$$y \times e^{x} = \int (\cos x - \sin x)\, e^{x}\, dx$$

$$= \int \cos x\, e^{x}\, dx - \int \sin x\, e^{x}\, dx$$

$$= \cos x\, e^{x} + \int (\sin x\, e^{x})\, dx - \int \sin x\, e^{x}\, dx + c$$

$$\boxed{y \times e^{x} = e^{x}\cos x + c}$$

13.i) $x\frac{dy}{dx} + y = x\log x$

$$\frac{dy}{dx} + \frac{y}{x} = \log x$$

$$P = \frac{1}{x} \quad Q = \log x$$

$$I.F = e^{\int P\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$$

$$yx = \int x\log x\,dx$$

$$= \log x\left[\frac{x^2}{2}\right] - \int \frac{1}{x} x \frac{x^2}{2}\,dx$$

$$\frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx$$

$$yx = \frac{x^2}{2}\log x - \frac{1}{2}\frac{x^2}{2} + c$$

$$\boxed{xy = \frac{x^2}{2}\log x - \frac{x^2}{4} + c}$$

ii) $x\frac{dy}{dx} + 2y = x^2 \log x$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{x^2}{x}\log x$$

$$P = \frac{2}{x} \qquad Q = x\log x$$

$$I.F = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

$$y \times x^2 = \int x \log x \; x^2\,dx$$

$$x^2 y = \int x^3 \log x\,dx$$

$$x^2y = \int x^3 \log x \, dx$$

$$x^2y = \log x \frac{x^4}{4} - \int \frac{1}{x} \times \frac{x^4}{4} \, dx$$

$$= \frac{x^4}{4} \log x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4}{4} \log x - \frac{1}{4} \frac{x^4}{4} + c$$

$$\boxed{x^2y = \frac{x^4}{4} \log x - \frac{x^4}{16} + c}$$

14i) $\frac{dy}{dx} + y\cot x = 2\cos x$

$P = \cot x \qquad Q = 2\cos x$

$\sin x = t$

$\cos x\, dx = dt$

$$I.F = e^{\int P\,dx} = e^{\int \cot x\,dx} = e^{\log \sin x} = \sin x$$

$$y \times \sin x = \int 2\cos x \sin x\, dx$$

$\sin 2x = 2\sin x \cos x$

$$y\sin x = 2\int t\,dt = \cancel{2}\frac{t^2}{\cancel{2}} + c$$

$$\boxed{y\sin x = \frac{\sin^2 x}{2} + c}$$

ii) $\frac{dy}{dx} - y\tan x = 2\sin x$

$P = -\tan x \quad Q = 2\sin x$

$$I.F = e^{\int P\,dx} = e^{-\int \tan x\,dx} = e^{\log \cos x} = \cos x$$

$$y\cos x = \int 2\sin x \cos x\,dx$$

$$y\cos x = 2\left[\frac{\sin^2 x}{2}\right] + c$$

$$\boxed{y\cos x = \sin^2 x + c}$$

15 i) $(\tan x)\frac{dy}{dx} + 2y = \sec x$

$$\frac{dy}{dx} + \frac{2y}{\tan x} = \frac{\sec x}{\tan x}$$

$$Q = \frac{\sec x}{\tan x} \Rightarrow \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \operatorname{cosec} x$$

$$P = \frac{2}{\tan x} = 2\cot x$$

$$I.F = e^{\int P\,dx} = e^{2\int \cot x\,dx} = e^{2\log \sin x} = e^{\log \sin^2 x} = \sin^2 x$$

$$y \sin^2 x = \int \operatorname{cosec} x \sin^2 x\,dx$$

$$= \int \frac{1}{\sin x} \times \sin^2 x\,dx$$

$$y \sin^2 x = \int \sin x \, dx$$

$$\boxed{y \sin^2 x = -\cos x + c}$$

ii) $\frac{dy}{dx} = x(x^2 - 2y)dx$

$\frac{dy}{dx} = x^3 - 2xy$

$\frac{dy}{dx} + 2xy = x^3$

$P = 2x \quad Q = x^3$

$I.F = e^{\int P dx} = e^{\int 2x\,dx} = e^{2\frac{x^2}{2}} = e^{x^2}$

$y e^{x^2} = \int x^3 \cdot e^{x^2} dx$

$y e^{x^2} = \int x^2 \; e^{x^2} \; x\,dx$

$x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$y e^{x^2} = \int t \; e^t \frac{dt}{2}$

$= \frac{1}{2}\left[t e^t - \int e^t dt\right]$

$y e^x = \frac{1}{2}\left[t e^t - e^t\right] + c$

$$\boxed{y e^x = \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2}\right] + c}$$